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Dual generalized Bernstein basis

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Abstract

The generalized Bernstein basis in the space Π_n of polynomials of degree at most n, being an extension of the q-Bernstein basis introduced by Philips [Bernstein polynomials based on the q-integers, Ann. Numer. Math. 4 (1997) 511–518], is given by the formula [S. Lewanowicz, P. Woźny, Generalized Bernstein polynomials, BIT Numer. Math. 44 (2004) 63–78]

$$B_i^n(x;\omega|\,q) := \frac{1}{(\omega;q)_n}\begin{bmatrix} n \\ i \end{bmatrix}_q x^i(\omega x^{-1};q)_i(x;q)_{n-i} \quad (i=0,1,\ldots,n).$$

We give explicitly the dual basis functions $D_k^n(x;a,b,\omega|\,q)$ for the polynomials $B_i^n(x;\omega|\,q)$, in terms of big q-Jacobi polynomials $P_k(x;a,b,\omega/q;q)$, a and b being parameters; the connection coefficients are evaluations of the q-Hahn polynomials. An inverse formula—relating big q-Jacobi, dual generalized Bernstein, and dual q-Hahn polynomials—is also given. Further, an alternative formula is given, representing the dual polynomial D_j^n $(0 \leqslant j \leqslant n)$ as a linear combination of $\min(j, n-j)+1$ big q-Jacobi polynomials with shifted parameters and argument. Finally, we give a recurrence relation satisfied by D_k^n, as well as an identity which may be seen as an analogue of the extended Marsden's identity [R.N. Goldman, Dual polynomial bases, J. Approx. Theory 79 (1994) 311–346].

Keywords: Generalized Bernstein basis; q-Bernstein basis; Bernstein basis; Discrete Bernstein basis; Dual basis; Big q-Jacobi polynomials; Little q-Jacobi polynomials; Shifted Jacobi polynomials

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1. Introduction

The *generalized Bernstein basis polynomials of degree n* ($n \in \mathbb{N}$) are defined by [16]

$$B_i^n(x;\omega|q) := \frac{1}{(\omega;q)_n} \begin{bmatrix} n \\ i \end{bmatrix}_q x^i (\omega x^{-1};q)_i (x;q)_{n-i} \quad (i = 0, 1, \ldots, n),$$

where q and ω are real parameters such that $q \neq 1$, and $\omega \neq 1, q^{-1}, \ldots, q^{1-n}$. Here we use the *q-Pochhammer symbol* defined for any $c \in \mathbb{C}$ by

$$(c;q)_0 := 1, \quad (c;q)_k := \prod_{j=0}^{k-1} (1 - cq^j) \quad (k \geqslant 1),$$

and the *q-binomial coefficient* given by

$$\begin{bmatrix} n \\ i \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_i (q;q)_{n-i}}.$$

For convenience we shall always assume that $q \in (0, 1)$. It should be stressed that the polynomials $B_i^n(x;\omega|q)$ are obtained as an extension of the *q-Bernstein basis polynomials*

$$b_i^n(x;q) = \begin{bmatrix} n \\ i \end{bmatrix}_q x^i (x;q)_{n-i} \quad (0 \leqslant i \leqslant n),$$

introduced by Phillips [20] (see also [18,19]). Notice that the classical *Bernstein basis polynomials* (see, e.g., [3, p. 66]),

$$B_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i} \quad (0 \leqslant i \leqslant n),$$

as well the *discrete Bernstein basis polynomials* [22,23],

$$b_i^n(N, x) = \frac{1}{(-N)_n} \binom{n}{i} (-x)_i (x-N)_{n-i} \quad (0 \leqslant i \leqslant n \leqslant N;\ N \in \mathbb{N}),$$

are also limiting forms of the polynomials $B_i^n(x;\omega|q)$; see (3.1)–(3.3), below. Here $(c)_k$ is the *Pochhammer symbol*, defined for any $c \in \mathbb{C}$ by

$$(c)_0 := 1, \quad (c)_k := c(c+1)\cdots(c+k-1) \quad (k \geqslant 1).$$

We define the *generalized Bézier curve* as the parametric curve

$$P_n^{\omega,q}(t) = \sum_{i=0}^{n} W_i B_i^n(u;\omega|q) \quad (u = \omega + (1-\omega)t;\ 0 \leqslant t \leqslant 1),$$

where $W_i \in \mathbb{R}^d$ ($1 \leqslant d \leqslant 3, i = 0, 1, \ldots, n$) are given points. This representation, being an extension of previously defined Bézier curve [6, Chapter 4] and q-Bézier curve [18], is advantageous for computations, on account of its shape-preserving property, and the numerical stability of the related de Casteljau algorithm for curve evaluation (see [16]).

In computer-aided geometric design it is often necessary to obtain polynomial approximations to more complicated functions in the Bernstein basis representation. In case of the least-squares approximation, the question of transformations between Bernstein and orthogonal bases arises in a natural way (see, e.g., [8,11,23]). Now, let us introduce the inner product $\langle \cdot, \cdot \rangle_{bqJ}$ in the space Π_n of polynomials of degree $\leqslant n$ by

$$\langle f, g \rangle_{bqJ} := \int_{\omega}^{aq} \frac{(x/a, qx/\omega; q)_\infty}{(x, bqx/\omega; q)_\infty} f(x)g(x)\, d_q x \quad (0 < aq,\ bq < 1, \omega < 0),$$

a and b being parameters. Notice that *big q-Jacobi polynomials* $P_k(x) \equiv P_k(x; a, b, \omega/q; q)$ (see (3.4)) form a set of orthogonal polynomials with respect to that inner product (see, e.g., [13, § 3.5]). The notation used is explained in the final part of this section. Associated with the basis $B_i^n(x; \omega|q)$, there is a unique *dual basis*

$$D_0^n(x; a, b, \omega|q), D_1^n(x; a, b, \omega|q), \dots, D_n^n(x; a, b, \omega|q) \in \Pi_n$$

defined so that

$$\left\langle D_i^n, B_j^n \right\rangle_{bqJ} = \delta_{ij} \quad (i, j = 0, 1, \dots, n).$$

Obviously, any polynomial $p \in \Pi_n$ can be written in the form

$$p = \sum_{k=0}^{n} \langle p, D_k^n \rangle_{bqJ} B_k^n.$$

However, little is known about the dual Bernstein bases; only in the classical Bernstein case, we have a recurrence relation given by Ciesielski [5], and a representation in terms of Bernstein basis polynomials obtained by Jüttler [11].

In this paper, we express explicitly the dual basis D_k^n in terms of big q-Jacobi polynomial basis P_k; the connection coefficients are evaluations of another orthogonal polynomials, namely q-Hahn polynomials. Hence, the polynomials D_k^n may be efficiently evaluated using the Clenshaw's algorithm (see, e.g., [25, § 10.2]) with a cost depending linearly on n. An inverse formula—relating big q-Jacobi, dual generalized Bernstein, and dual q-Hahn polynomials—is also given. Further, an alternative formula is given, representing the dual polynomial D_j^n $(0 \leqslant j \leqslant n)$ as a "short" linear combination of $\min(j, n-j)+1$ big q-Jacobi polynomials with shifted parameters and argument. Finally, we give a recurrence relation satisfied by D_k^n, as well as an identity which may be seen as an analogue of the extended Marsden's identity [10].

In Section 2, we give some general results on dual bases. Section 3 contains the above-mentioned results for the most general case, i.e. dual basis for generalized Bernstein basis. For the reader's convenience, in Sections 4 and 5, we discuss separately dual bases for q-Bernstein and classical Bernstein polynomial bases. In Section 6, we give some examples of possible applications of the presented results, in particular—the recurrence relation given for the dual Bernstein basis, to obtain a least-squares polynomial approximation with a prescribed accuracy.

We end this section with a list of notation and terminology used in the paper. For more details the reader is referred to the monographs [1] by G. Andrews, R. Askey and R. Roy, or [9] by G. Gasper and M. Rahman, or the report [13] by R. Koekoek and R. Swarttouw. The *q-integral* is defined by

$$\int_a^b f(x)\, d_q x := \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x,$$

where

$$\int_0^a f(x)\, d_q x := a(1-q)\sum_{k=0}^{\infty} f(aq^k)q^k.$$

Extending the meaning of the q-Pochhammer symbol, put

$$(x;q)_\infty := \prod_{j=0}^{\infty}(1-q^j x).$$

In the sequel, we make use of the convention

$$(c_1, c_2, \ldots, c_k)_n := \prod_{j=1}^{k}(c_j)_n, \quad (c_1, c_2, \ldots, c_k; q)_n := \prod_{j=1}^{k}(c_j; q)_n.$$

For $c \in \mathbb{C}$, we define the *q-number* $[c]_q$ by

$$[c]_q := \frac{q^c - 1}{q - 1}.$$

The *generalized hypergeometric series* is defined by (see, e.g., [1, § 2.1])

$${}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \middle| z\right) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r)_k}{(1, b_1, \ldots, b_s)_k} z^k,$$

while the *basic hypergeometric series* is defined by (see, e.g., [1, § 10.9])

$${}_r\phi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \middle| q; z\right) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(q, b_1, \ldots, b_s; q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k,$$

where $r, s \in \mathbb{Z}_+$ and $a_1, \ldots, a_r, b_1, \ldots, b_s, z \in \mathbb{C}$.

2. General results on dual polynomial bases

2.1. Orthogonal basis and duality

Let Π_n denote the linear space of all polynomials of degree $\leqslant n$. Let $\{P_k\}$ be a sequence of orthogonal polynomials with respect to a given inner product $\langle \cdot, \cdot \rangle$ in Π_n, i.e. $\deg P_k = k$ $(k = 0, 1, \ldots, n)$ and

$$\langle P_i, P_j \rangle = h_i \delta_{ij} \quad (h_i > 0;\, i, j = 0, 1, \ldots, n).$$

Let $b_0, b_1, \ldots, b_n$ be a basis in Π_n. There exists a uniquely defined basis $d_0, d_1, \ldots, d_n \in \Pi_n$, called *dual basis*, such that

$$\langle d_i, b_j \rangle = \delta_{ij} \quad (i, j = 0, 1, \ldots, n).$$

Notice that any polynomial $w \in \Pi_n$ can be written in the form

$$w = \sum_{k=0}^{n} \langle w, d_k \rangle b_k.$$

Lemma 2.1. *Let α_{ik} and β_{jk} be the coefficients in*

$$P_k = \sum_{i=0}^{n} \alpha_{ki} b_i, \quad b_k = \sum_{i=0}^{n} \beta_{ki} P_i \quad (0 \leqslant k \leqslant n). \tag{2.1}$$

Then the dual basis satisfies

$$P_i = h_i \sum_{k=0}^{n} \beta_{ki} d_k, \quad d_i = \sum_{k=0}^{n} h_k^{-1} \alpha_{ki} P_k \quad (0 \leqslant i \leqslant n). \tag{2.2}$$

Proof. From Eq. (2.1), we deduce

$$\sum_{k=0}^{n} \alpha_{ik} \beta_{kj} = \delta_{ij} = \sum_{k=0}^{n} \alpha_{ki} \beta_{jk} \quad (i, j = 0, 1, \ldots, n). \tag{2.3}$$

We have to determine coefficients μ_{ik} and v_{ik} such that

$$P_j = \sum_{k=0}^{n} \mu_{jk} d_k, \quad d_i = \sum_{k=0}^{n} v_{ik} P_k.$$

Using the first equations of (2.1) and (2.2), we obtain

$$h_j \delta_{ij} = \langle P_i, P_j \rangle = \sum_{k=0}^{n} \alpha_{ik} \sum_{m=0}^{n} \mu_{jm} \langle b_k, d_m \rangle = \sum_{k=0}^{n} \alpha_{ik} \mu_{jk},$$

which, in view of the first equality (2.3), means that $\mu_{jk} = h_j \beta_{kj}$.

Similarly, using the second equalities of (2.1), (2.2), and of (2.3), we find that $v_{ik} = h_k^{-1} \alpha_{ki}$. □

2.2. An identity

It is known (see, e.g., [1, p. 246]) that the *Christoffel–Darboux kernel*

$$K_n(x, t) := \sum_{k=0}^{n} h_k^{-1} P_k(t) P_k(x)$$

can be expressed as

$$K_n(x, t) = \frac{\varkappa_n}{\varkappa_{n+1} h_n} \frac{P_{n+1}(x) P_n(t) - P_{n+1}(t) P_n(x)}{x - t},$$

where $\varkappa_m$ is the leading coefficient of the polynomial P_m $(m = 0, 1, \ldots)$.

Lemma 2.2. *The following identity holds*:

$$\sum_{i=0}^{n} b_i(x) d_i(t) = K_n(x, t). \tag{2.4}$$

Proof. Making use of the second equations of (2.2), (2.1) and (2.3), we obtain

$$\sum_{i=0}^{n} d_i(t)b_i(x) = \sum_{i=0}^{n}\left(\sum_{k=0}^{n} h_k^{-1}\alpha_{ki}P_k(t)\right)\left(\sum_{l=0}^{n}\beta_{il}P_l(x)\right)$$
$$= \sum_{k=0}^{n} h_k^{-1}P_k(t)\sum_{l=0}^{n}P_l(x)\sum_{i=0}^{n}\alpha_{ki}\beta_{il} = \sum_{k=0}^{n} h_k^{-1}P_k(t)P_k(x). \qquad \square$$

Eq. (2.4) may be considered as an analogue of the extended Marsden's identity in an alternative theory of dual polynomial bases discussed in [10].

2.3. *Representation of polynomials*

Let w be a polynomial of degree n. Assume that the coefficients a_k in

$$w(x) = \sum_{k=0}^{n} a_k P_k(x)$$

are known. In some applications (see, e.g., [7,8]), there is a need to transform the above expansion to

$$w(x) = \sum_{k=0}^{n} c_k b_k(x).$$

Now, it is easy to observe that

$$c_k = \sum_{i=0}^{n} \alpha_{ik}a_i \quad (k = 0, 1, \ldots, n), \tag{2.5}$$

and that the inverse transformation is given by

$$a_k = \sum_{j=0}^{n} \beta_{jk}c_j \quad (k = 0, 1, \ldots, n).$$

3. Dual generalized Bernstein basis

3.1. *Generalized Bernstein basis*

In [16], we have introduced the *generalized Bernstein basis polynomials of degree n* ($n\in\mathbb{N}$) by

$$B_i^n(x;\omega|q) := \frac{1}{(\omega;q)_n}\begin{bmatrix} n \\ i \end{bmatrix}_q x^i(\omega x^{-1};q)_i(x;q)_{n-i} \quad (0\leqslant i\leqslant n),$$

where q and ω are real parameters such that $q \neq 1$, and $\omega \neq 1, q^{-1}, \ldots, q^{1-n}$. Notice that the classical *Bernstein basis polynomials* (see, e.g., [3, p. 66])

$$B_i^n(x) = \binom{n}{i} x^i(1-x)^{n-i} \quad (0\leqslant i\leqslant n),$$

the *discrete Bernstein basis polynomials* [22,23]

$$b_i^n(N,x) = \frac{1}{(-N)_n}\binom{n}{i}(-x)_i(x-N)_{n-i} \quad (0\leqslant i\leqslant n\leqslant N;\ N\in\mathbb{N}),$$

as well as the *q-Bernstein basis polynomials*

$$b_i^n(x;q) = \begin{bmatrix} n \\ i \end{bmatrix}_q x^i (x;q)_{n-i} \quad (0\leqslant i\leqslant n),$$

recently introduced by Phillips (see [18–20]), are limiting forms of the polynomials $B_i^n(x;\omega|q)$. Namely, we have

$$\lim_{q\uparrow 1} B_i^n(x;\omega|q) = B_i^n\left(\frac{x-\omega}{1-\omega}\right), \tag{3.1}$$

$$\lim_{q\uparrow 1} B_i^n(q^{-x};q^{-N}|q) = b_{n-i}^n(N,x), \tag{3.2}$$

$$B_i^n(x;0|q) = b_i^n(x;q). \tag{3.3}$$

Introduce the inner product in the space Π_n of polynomials of degree $\leqslant n$ by

$$\langle f,g\rangle_{bqJ} := \int_\omega^{aq} \frac{(x/a, qx/\omega;q)_\infty}{(x,bqx/\omega)_\infty} f(x)g(x)\,d_qx,$$

a and b being parameters, $0<aq,\ bq<1, \omega<0$. Notice that *big q-Jacobi polynomials* (see, e.g., [13, § 3.5])

$$P_k(x) \equiv P_k(x;a,b,\omega/q;q) := {}_3\phi_2\left(\left.\begin{matrix} q^{-k}, abq^{k+1}, x \\ aq, \omega \end{matrix}\,\right|\, q;q\right) \tag{3.4}$$

form a set of orthogonal polynomials with respect to that inner product, i.e.,

$$\langle P_k, P_l\rangle_{bqJ} = h_k\delta_{kl}, \tag{3.5}$$

where

$$h_k := M\frac{(1-\sigma)}{(1-\sigma q^{2k})}\frac{(q,bq,\sigma q/\omega;q)_k}{(aq,\sigma,\omega;q)_k}(-a\omega q)^k q^{\binom{k}{2}}, \tag{3.6}$$

and

$$M := aq(1-q)\frac{(q,\sigma q,\omega/(aq),aq^2/\omega;q)_\infty}{(aq,bq,\omega,\sigma q/\omega;q)_\infty}, \quad \sigma := abq. \tag{3.7}$$

Obviously, $P_0, P_1, \dots, P_n$ also form a basis in Π_n. In [16], we obtained the following results. Recall that the *q-Hahn polynomials* are given by (see, e.g., [9, Eq. (7.3.21)], or [13, § 3.6])

$$Q_k(q^{-x};a,b,N|q) := {}_3\phi_2\left(\left.\begin{matrix} q^{-k}, abq^{k+1}, q^{-x} \\ aq, q^{-N} \end{matrix}\,\right|\, q;q\right), \tag{3.8}$$

while the *dual q-Hahn polynomials* are defined by [13, § 3.7]

$$R_k(\mu(x);\gamma,\delta,N|q) := {}_3\phi_2\left(\left.\begin{matrix} q^{-k}, q^{-x}, \gamma\delta q^{x+1} \\ \gamma q, q^{-N} \end{matrix}\,\right|\, q;q\right), \tag{3.9}$$

where $0\leqslant k\leqslant N$, $N\in\mathbb{N}$, and $\mu(x) := q^{-x}+\gamma\delta q^{x+1}$. Generalized Bernstein basis polynomials have the following representation in terms of big q-Jacobi polynomial basis [16]:

$$B^n_{n-i}(x;\omega|q) = q^{\binom{i}{2}}(aq)^n \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{(bq;q)_n}{(\sigma q;q)_n}(-\sigma q^n)^{-i}\frac{(aq;q)_i}{(b^{-1}q^{-n};q)_i} \times\sum_{j=0}^{n} q^{-\binom{j+1}{2}}\left(-\frac{q^n}{a}\right)^j \frac{1-\sigma q^{2j}}{1-\sigma}\frac{(aq,\sigma,q^{-n};q)_j}{(q,bq,\sigma q^{n+1};q)_j} \times Q_j\left(q^{-i};a,b,n\middle|q\right)P_j(x;a,b,\omega/q;q), \tag{3.10}$$

where $0\leqslant i\leqslant n$. Conversely, big q-Jacobi polynomials have the following representation in terms of generalized Bernstein basis [16]:

$$P_i(x;a,b,\omega/q;q) = \sum_{j=0}^{n} R_{n-j}\left(\mu(i);a,b,n|q\right)B^n_j(x;\omega|q), \tag{3.11}$$

where $\mu(i) := q^{-i}+\sigma q^i$, $0\leqslant i\leqslant n$. In particular, we have

$$P_n(x;a,b,\omega/q;q) = \sum_{j=0}^{n}\frac{(q^{-n}/b;q)_j}{(aq;q)_j}(\sigma q^n)^j B^n_{n-j}(x;\omega|q). \tag{3.12}$$

3.2. Dual generalized Bernstein basis polynomials

Associated with the generalized Bernstein basis, there is a unique *dual basis*

$$D^n_0(x;a,b,\omega|q),\ D^n_1(x;a,b,\omega|q),\ \ldots,\ D^n_n(x;a,b,\omega|q)\in\Pi_n$$

defined so that

$$\left\langle D^n_i, B^n_j\right\rangle_{bqJ} = \delta_{ij}\quad (i,j=0,1,\ldots,n).$$

We give a number of properties of the polynomials $D^n_i(x;a,b,\omega|q)$.

3.2.1. Recurrence relation

Theorem 3.1. *The following recurrence relation holds*:

$$D^{n+1}_i(x;a,b,\omega|q) = \gamma^n_i D^n_i(x;a,b,\omega|q)+(1-\gamma^n_i)D^n_{i-1}(x;a,b,\omega|q) + \vartheta^n_i P_{n+1}(x;a,b,\omega/q;q), \tag{3.13}$$

where $0\leqslant i\leqslant n+1$, $D^n_{-1}(x;a,b,\omega|q) = D^n_{n+1}(x;a,b,\omega|q) = 0$, *and*

$$\gamma^n_i := \frac{[n-i+1]_q}{[n+1]_q}, \tag{3.14}$$

$$\vartheta^n_i := h^{-1}_{n+1}\frac{(q^{-n-1}/b;q)_{n+1-i}}{(aq;q)_{n+1-i}}(\sigma q^{n+1})^{n+1-i}. \tag{3.15}$$

Proof. Argument used in the proof is fully analogous to the one used in [5] to prove similar result for the (classical) dual Bernstein basis polynomials, associated with the Legendre inner product (in this connection, see Theorem 5.1). Given a polynomial $p \in \Pi_n$, we have

$$p = \sum_{i=0}^{n} \langle p, D_i^n \rangle_{bqJ} B_i^n. \tag{3.16}$$

Using the identity [16]

$$B_i^n = \gamma_i^n B_i^{n+1} + \left(1 - \gamma_{i+1}^n\right) B_{i+1}^{n+1},$$

where the notation used is that of (3.14), we obtain

$$p = \sum_{i=0}^{n+1} \langle p, \gamma_i^n D_i^n + (1 - \gamma_i^n) D_{i-1}^n \rangle_{bqJ} B_i^{n+1}.$$

Comparing this result with a representation of p as a linear combination of $B_0^{n+1},\ B_1^{n+1}, \ldots,$ B_{n+1}^{n+1}, similar to (3.16), we obtain

$$\left\langle p, D_i^{n+1} \right\rangle_{bqJ} = \langle p, \gamma_i^n D_i^n + (1 - \gamma_i^n) D_{i-1}^n \rangle_{bqJ},$$

thus the polynomial

$$w := D_i^{n+1} - \gamma_i^n D_i^n - (1 - \gamma_i^n) D_{i-1}^n \in \Pi_{n+1}$$

must have the form $w = \vartheta_i^n P_{n+1}$, where (cf. (3.12))

$$\begin{aligned}
\vartheta_i^n h_{n+1} =& \langle w, P_{n+1} \rangle_{bqJ} = \left\langle D_i^{n+1}, P_{n+1} \right\rangle_{bqJ} \\
=& \sum_{j=0}^{n+1} \frac{(q^{-n-1}/b; q)_j}{(aq; q)_j} (\sigma q^{n+1})^j \left\langle D_i^{n+1}, B_{n+1-j}^{n+1} \right\rangle_{bqJ} \\
=& \frac{(q^{-n-1}/b; q)_{n+1-i}}{(aq; q)_{n+1-i}} (\sigma q^{n+1})^{n+1-i}.
\end{aligned}$$

Hence follows (3.13). □

3.2.2. Expansions

Theorem 3.2. *Big q-Jacobi polynomials have the following representation in terms of dual generalized Bernstein basis*:

$$\begin{aligned}
P_i\,(x; a, b, \omega/q; q) = M & \frac{(bq; q)_n}{(\sigma q; q)_n} \frac{(q^{-n}, \sigma q/\omega; q)_i}{(\sigma q^{n+1}, \omega; q)_i} \omega^i (aq^{i+1})^n \\
& \times \sum_{j=0}^{n} q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-\sigma q^n)^{-j} (aq; q)_j}{(b^{-1} q^{-n}; q)_j} \\
& \times R_j\big(\mu(i); a, b, n \big| q\big)\, D_{n-j}^n(x; a, b, \omega | q),
\end{aligned} \tag{3.17}$$

where $\mu(i) := q^{-i} + abq^{i+1}$, $0 \leqslant i \leqslant n$. *Dual generalized Bernstein basis polynomials have the following representation in terms of big* q-*Jacobi polynomial basis*:

$$D_j^n(x; a, b, \omega|q) = M^{-1} \sum_{k=0}^{n} \frac{1-\sigma q^{2k}}{1-\sigma} \frac{(aq, \sigma, \omega; q)_k}{(q, bq, \sigma q/\omega; q)_k} (-a\omega)^{-k} q^{-\binom{k+1}{2}} \times Q_k\left(q^{j-n}; a, b, n|q\right) P_k\left(x; a, b, \omega/q; q\right), \tag{3.18}$$

where $0 \leqslant j \leqslant n$.

Proof. Use Lemma 2.1, (3.10) and (3.11), and the relation (see, e.g., [13, p. 77])

$$Q_k(q^{-x}; a, b, N|q) = R_x(\mu(k); a, b, N|q). \qquad \square \tag{3.19}$$

Remark 3.3. By the identity [24]

$$Q_k(q^{-x}; A, B, N|q) = \frac{(q^{-k}/B; q)_k}{(Aq; q)_k} (ABq^{k+1})^k Q_k(q^{N-x}; B^{-1}, A^{-1}, N|q^{-1}),$$

Eq. (3.18) can be written as

$$D_j^n(x; a, b, \omega|q) = M^{-1} \sum_{k=0}^{n} \frac{1-\sigma q^{2k}}{1-\sigma} \frac{(\sigma, \omega; q)_k}{\omega^k (q, \sigma q/\omega; q)_k} \times Q_k(r^{-j}; b^{-1}, a^{-1}, n|r) P_k\left(x; a, b, \omega/q; q\right), \tag{3.20}$$

where $r := q^{-1}$, $0 \leqslant j \leqslant n$.

In the next theorem, we give alternative formulas, representing the dual polynomial $D_j^n(x; a, b, \omega|q)$ $(0 \leqslant j \leqslant n)$ as a linear combination of certain $\min(j, n-j)+1$ big q-Jacobi polynomials with shifted parameters and argument.

Theorem 3.4. *The following formulas hold for* $j = 0, 1, \ldots, n$:

$$D_j^n(x; a, b, \omega|q) = A_n \frac{(q^{-n}/a; q)_j}{(bq; q)_j} \sum_{k=0}^{j} (bq^{j+1})^k \frac{(q^{-j}, \omega q^{-n}/\sigma; q)_k}{(\omega q, q^{-n}/a; q)_k} \times P_{n-k}\left(x; a, bq^{k+1}, \omega q^k; q\right); \tag{3.21}$$

$$D_{n-j}^n(x; a, b, \omega|q) = B_n (\sigma q^n)^j \frac{(q^{-n}/b; q)_j}{(aq; q)_j} \sum_{k=0}^{j} (\sigma q^{n+2})^k \frac{(q^{-j}, \omega q^{-n}/\sigma; q)_k}{(aq^2, \omega q; q)_k} \times P_{n-k}\left(q^{k+1}x; aq^{k+1}, b, \omega q^k; q\right), \tag{3.22}$$

where

$$A_n := \frac{(\sigma q, \omega q; q)_n}{M\omega^n (q, \sigma q/\omega; q)_n}, \qquad B_n := q^{-\binom{n+1}{2}} \frac{(aq^2; q)_n}{(-aq)^n (bq; q)_n} A_n. \tag{3.23}$$

Proof. We will prove the formula (3.21). Given $j\in\{0,1,\ldots,n\}$, let us define for $m=0,1,\ldots,j$

$$S_m(x) := \sum_{k=0}^{n-m} \frac{(1-\sigma q^{2k+m})(\sigma q^m,\omega q^m;q)_k}{(\omega q^m)^k(1-\sigma q^m)(q,\sigma q/\omega;q)_k} q^{-\binom{k+1}{2}-2mk} \times Q_{k,m} P_k\left(x;a,bq^m,\omega q^{m-1};q\right) \tag{3.24}$$

with

$$Q_{k,m} := Q_k\left(r^{m-j};r^m/b,1/a,n-m|r\right), \quad r:=q^{-1}.$$

Notice that by (3.20) we have

$$D_j^n(x;a,b,\omega|q) = M^{-1}S_0(x). \tag{3.25}$$

We will show that (3.24) satisfies the following recurrence:

$$S_m(x) = f_m(x) + q^{j-n-1}\frac{(1-q^{m-j})(1-\sigma q^{m+1})}{a(1-bq^{m+1})(1-q^{m-n})} S_{m+1}(x) \tag{3.26}$$

for $0\leqslant m\leqslant j$, with the "end" value $S_{j+1}(x):=0$. Here

$$f_m(x) := \frac{(\omega q^m)^{m-n}(\sigma q,\omega q;q)_n(q^{-n}/a;q)_j(bq;q)_m}{(q,\sigma q/\omega;q)_{n-m}(\sigma q,\omega q;q)_m(bq,q^{-n}/a;q)_j} \times P_{n-m}\left(x;a,bq^{m+1},\omega q^m;q\right). \tag{3.27}$$

We shall need the identity

$$\begin{aligned}(1-ABq^{2k+1})&(1-Cq)P_k(x;A,B,C;q)\\ &=(1-ABq^{k+1})(1-Cq^{k+1})P_k(x;A,Bq,Cq;q)\\ &\quad -Cq(1-q^k)(1-ABq^k/C)P_{k-1}(x;A,Bq,Cq;q),\end{aligned} \tag{3.28}$$

which can be verified by substituting

$$P_k(x;A,B,C;q) = (-C)^k q^{\binom{k+1}{2}}\frac{(ABq/C;q)_k}{(Cq;q)_k}\,{}_3\phi_2\left(\begin{matrix}q^{-k},ABq^{k+1},Aq/x\\ Aq,ABq/C\end{matrix}\,\middle|\, q;\frac{x}{C}\right);$$

the last equation can be obtained by applying the transformation [9, Eq. (3.2.2)] to the standard form

$$P_k(x;A,B,C;q) = {}_3\phi_2\left(\begin{matrix}q^{-k},ABq^{k+1},x\\ Aq,Cq\end{matrix}\,\middle|\, q;q\right).$$

Using (3.28) with $A = a$, $B = bq^m$ and $C = \omega q^{m-1}$ in (3.24), we get

$$\begin{aligned}
S_m(x) &= \sum_{k=0}^{n-m} \frac{(\sigma q^{m+1}, \omega q^{m+1}; q)_k}{(\omega q^m)^k (q, \sigma q/\omega; q)_k} Q_{k,m} P_k\left(x; a, bq^{m+1}, \omega q^m; q\right) \\
&\quad - \sum_{k=1}^{n-m} \frac{(\sigma q^{m+1}, \omega q^{m+1}; q)_{k-1}}{(\omega q^m)^{k-1} (q, \sigma q/\omega; q)_{k-1}} Q_{k,m} P_{k-1}\left(x; a, bq^{m+1}, \omega q^m; q\right) \\
&= f_m(x) + \sum_{k=0}^{n-m-1} \frac{(\sigma q^{m+1}, \omega q^{m+1}; q)_k}{(\omega q^m)^k (q, \sigma q/\omega; q)_k} \\
&\quad \times \left(Q_{k,m} - Q_{k+1,m}\right) P_k\left(x; a, bq^{m+1}, \omega q^m; q\right) \\
&= f_m(x) + q^{j-n} \frac{1 - q^{m-j}}{a(1 - bq^{m+1})(1 - q^{m-n})} \\
&\quad \times \sum_{k=0}^{n-m-1} \frac{(1 - \sigma q^{2k+m+1})(\sigma q^{m+1}, \omega q^{m+1}; q)_k}{(\omega q^{m+1})^k (q, \sigma q/\omega; q)_k} \\
&\quad \times Q_{k,m+1} P_k\left(x; a, bq^{m+1}, \omega q^m; q\right) \\
&= f_m(x) + q^{j-n} \frac{(1 - q^{m-j})(1 - \sigma q^{m+1})}{a(1 - bq^{m+1})(1 - q^{m-n})} S_{m+1}(x),
\end{aligned}$$

where

$$f_m(x) := \frac{(\sigma q^{m+1}, \omega q^{m+1}; q)_{n-m}}{(\omega q^m)^{n-m} (q, \sigma q/\omega; q)_{n-m}} Q_{n-m,m} P_{n-m}\left(x; a, bq^{m+1}, \omega q^m; q\right). \tag{3.29}$$

In the last but one leg we used the identity

$$\begin{aligned}
&Q_k\left(q^{-t}; \gamma, \delta, N|q\right) - Q_{k+1}\left(q^{-t}; \gamma, \delta, N|q\right) \\
&\quad = \frac{(1 - q^{-t})(1 - \gamma\delta q^{2k+2})}{q^k (1 - \gamma q)(1 - q^{-N})} Q_k(q^{1-t}; \gamma q, \delta, N - 1|q),
\end{aligned}$$

which can be deduced from [13, Eq. (3.7.6)] by the duality principle (3.19). Using

$$\begin{aligned}
Q_{n-m,m} &= Q_{n-m}\left(r^{m-j}, \frac{r^m}{b}, \frac{1}{a}, n - m \middle| r\right) = {}_2\phi_1\left(\begin{matrix} r^{m-j}, r^{n+1}/(ab) \\ r^{m+1}/b \end{matrix} \middle| r; r\right) \\
&= \left(\frac{r^{n+1}}{ab}\right)^{j-m} \frac{(ar^{m-n}; r)_{j-m}}{(r^{m+1}/b; r)_{j-m}} = \frac{(q^{-n}/a; q)_j (bq; q)_m}{(bq; q)_j (q^{-n}/a; q)_m}
\end{aligned} \tag{3.30}$$

(see, e.g., [13, Eq. (0.5.9)]), we readily establish that $f_m(x)$ defined in (3.29) can be written in the form (3.27).

This concludes the proof of the recurrence (3.26). Thus,

$$S_0(x) = \sum_{k=0}^{j} f_k(x) \prod_{m=0}^{k-1} q^{j-n} \frac{(1 - q^{m-j})(1 - \sigma q^{m+1})}{a(1 - bq^{m+1})(1 - q^{m-n})}.$$

Inserting (3.27), simplifying and remembering (3.25), we obtain (3.21).

The proof of (3.22), which we do not give here, goes along the same lines. □

In particular, Theorem 3.4 implies the following formulas:

$$D_0^n(x;a,b,\omega|q) = A_n P_n(x;a,bq,\omega;q);$$

$$D_1^n(x;a,b,\omega|q) = A_n \frac{1-q^{-n}/a}{1-bq}\Big[P_n(x;a,bq,\omega;q) + \frac{(1-q)(\sigma q^n-\omega)}{(1-\omega q)(1-aq^n)} P_{n-1}\left(x;a,bq^2,\omega q;q\right)\Big];$$

$$D_{n-1}^n(x;a,b,\omega|q) = B_n aq \frac{1-bq^n}{aq-1}\Big[P_n(qx;aq,b,\omega;q) + q\frac{(1-q)(\omega-\sigma q^n)}{(1-aq^2)(1-\omega q)} P_{n-1}(q^2x;aq^2,b,\omega q;q)\Big];$$

$$D_n^n(x;a,b,\omega|q) = B_n P_n(qx;aq,b,\omega;q),$$

where the notation used is that of (3.23).

3.2.3. An identity

Theorem 3.5. *The following identity holds*:

$$\sum_{i=0}^{n} B_i^n(x;\omega|q) D_i^n(t;a,b,\omega|q) = L_n \frac{P_{n+1}(x)P_n(t) - P_{n+1}(t)P_n(x)}{x-t}, \tag{3.31}$$

where

$$L_n := \frac{(-a\omega)^{-n} q^{-\binom{n+1}{2}}}{M(1-\sigma)(1-\sigma q^{2n+1})} \frac{(aq,\sigma,\omega;q)_{n+1}}{(q,bq,\sigma q/\omega;q)_n},$$

and where we used the notation of (3.4) *and* (3.7).

Proof. The thesis follows by application of Lemma 2.2. □

4. Dual q-Bernstein basis

4.1. q-Bernstein basis polynomials

The *q-Bernstein basis polynomials of degree n* ($n \in \mathbb{N}$),

$$b_i^n(x;q) = \begin{bmatrix} n \\ i \end{bmatrix}_q x^i (x;q)_{n-i} \quad (0 \leqslant i \leqslant n),$$

are recently introduced by Phillips [20] (see also [18,19]). Remember that *little q-Jacobi polynomials* are given by

$$p_k(x;a,b|q) := {}_2\phi_1\left(\begin{matrix} q^{-k}, abq^{k+1} \\ aq \end{matrix} \,\middle|\, q; qx\right) \quad (k = 0, 1, \ldots)$$

(see, e.g., [13, § 3.12]). Let us introduce the *shifted little q-Jacobi polynomials* by

$$p_k^*(x; a, b|q) := p_k(x/(bq); a, b|q) \quad (0 < aq < 1, bq < 1). \tag{4.1}$$

Polynomials (4.1) are orthogonal with respect to the inner product

$$\langle f, g\rangle_{lqJ} := \int_0^1 x^\alpha \frac{(qx; q)_\infty}{(bqx; q)_\infty} f(bqx)g(bqx)\, d_qx,$$

where $a = q^\alpha$, $b = q^\beta$, $\alpha, \beta > -1$. More specifically,

$$\langle p_k^*, p_l^*\rangle_{lqJ} = h_k\delta_{kl}, \tag{4.2}$$

where

$$h_k := C(aq)^k \frac{1-\sigma}{1-\sigma q^{2k}} \frac{(q, bq; q)_k}{(aq, \sigma; q)_k} \quad (k \geqslant 0) \tag{4.3}$$

and

$$C \equiv C(a, b) := (1-q)\frac{(q, \sigma q; q)_\infty}{(aq, bq; q)_\infty}, \quad \sigma := abq. \tag{4.4}$$

See, e.g., [13, § 3.12]. Notice that [13, § 3.5]

$$P_k(x; b, a, 0; q) = (-bq)^k q^{\binom{k}{2}} \frac{(aq; q)_k}{(bq; q)_k} p_k^*(x; a, b|q). \tag{4.5}$$

Koornwinder [14] has shown that the orthogonality relation (4.2) is a limit case of (3.5) when $\omega \uparrow 0$.

Let us recall the following results [2]. q-Bernstein polynomials have the following representation in terms of the shifted little q-Jacobi polynomial basis:

$$\begin{aligned} b_{n-i}^n(x; q) = {} & (-\sigma q^n)^{-i} q^{\binom{i}{2}} (bq)^n \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{(aq; q)_n}{(\sigma q; q)_n} \frac{(bq; q)_i}{(a^{-1}q^{-n}; q)_i} \\ & \times \sum_{j=0}^n q^{nj} \frac{(1-\sigma q^{2j})(\sigma, q^{-n}; q)_j}{(1-\sigma)(q, \sigma q^{n+1}; q)_j} \\ & \times Q_j\left(q^{-i}; b, a, n\middle|q\right) p_j^*(x; a, b|q), \end{aligned} \tag{4.6}$$

where $0 \leqslant i \leqslant n$, and the notation used is that of (3.8). Shifted little q-Jacobi polynomials have the following representation in terms of q-Bernstein basis:

$$p_i^*(x; a, b|q) = (-bq)^{-i} q^{-\binom{i}{2}} \frac{(bq; q)_i}{(aq; q)_i} \sum_{j=0}^n R_{n-j}\left(\mu(i); b, a, n|q\right) b_j^n(x; q), \tag{4.7}$$

where $\mu(i) := q^{-i} + \sigma q^i$, $0 \leqslant i \leqslant n$, and the notation used is that of (3.9).

4.2. Dual q-Bernstein basis polynomials

Dual q-Bernstein basis polynomials of nth degree,

$$d_0^n(x; a, b; q),\ d_1^n(x; a, b; q), \ldots, d_n^n(x; a, b; q),$$

are defined so that $\left\langle d_i^n, b_j^n \right\rangle_{lqJ} = \delta_{ij}$ $(i, j = 0, 1, \ldots, n)$, where $b_j^n = b_j^n(\cdot; q)$, $d_j^n = d_j^n(\cdot; a, b; q)$. In view of the above cited Koornwinder's observation and (3.3), we must have

$$d_i^n(x; a, b; q) = D_i^n(x; b, a, 0|q) \quad (0 \leqslant i \leqslant n). \tag{4.8}$$

4.2.1. Recurrence relation

Corollary 4.1. *We have the recurrence relation*

$$\begin{aligned} d_i^{n+1}(x; a, b; q) = {} & \gamma_i^n\, d_i^n(x; a, b; q) + (1 - \gamma_i^n)\, d_{i-1}^n(x; a, b; q) \\ & + \vartheta_i^n\, p_{n+1}^*(x; a, b|q), \end{aligned} \tag{4.9}$$

where $0 \leqslant i \leqslant n + 1$, $d_{-1}^n(x; a, b; q) = d_{n+1}^n(x; a, b; q) = 0$, *and*

$$\gamma_i^n := \frac{[n - i + 1]_q}{[n + 1]_q}, \qquad \vartheta_i^n := h_{n+1}^{-1} \frac{(q^{-n-1}/b; q)_i}{(aq; q)_i},$$

notation used being that of (4.3).

Proof. The thesis follows by Theorem 3.1, in view of (4.8) and (4.5). □

4.2.2. Expansions

Corollary 4.2. *Shifted little q-Jacobi polynomials have the following representation in terms of dual q-Bernstein basis*:

$$\begin{aligned} p_i^*(x; a, b|q) = {} & C \frac{(aq; q)_n}{(\sigma q; q)_n} (bq)^n \frac{(bq, q^{-n}; q)_i}{(aq, \sigma q^{n+1}; q)_i} (aq^{n+1})^i \\ & \times \sum_{k=0}^{n} (-\sigma q^n)^{-k} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(bq; q)_k}{(a^{-1}q^{-n}; q)_k} \\ & \times R_k\big(\mu(i); b, a, n\big|q\big)\, d_{n-k}^n(x; a, b; q), \end{aligned} \tag{4.10}$$

where $\mu(i) := q^{-i} + \sigma q^i$, $0 \leqslant i \leqslant n$. *Dual q-Bernstein basis polynomials have the following representation in terms of shifted little q-Jacobi polynomial basis*:

$$\begin{aligned} d_j^n(x; a, b; q) = {} & C^{-1} \sum_{k=0}^{n} (-1)^k q^{-\binom{k+1}{2}} \frac{(1 - \sigma q^{2k})}{(1 - \sigma)\sigma^k} \frac{(\sigma; q)_k}{(q; q)_k} \\ & \times Q_k(q^{j-n}; b, a, n|q)\, p_k^*(x; a, b|q), \end{aligned} \tag{4.11}$$

where $0 \leqslant j \leqslant n$.

Proof. The result follows by reversing the roles of a and b in (3.17), (3.18), letting $\omega \uparrow 0$ and then using (4.5) and (4.8). □

Remark 4.3. Alternatively, we can obtain the above corollary using Lemma 2.1, (4.6), (4.7), and (3.19).

Theorem 3.4 implies the following results.

Corollary 4.4. *The following formulas hold for* $j = 0, 1, \ldots, n$:

$$d_j^n(x; a, b; q) = A_n \frac{(q^{-n}/b; q)_j}{(aq; q)_j} \times \sum_{k=0}^{j} (aq^{j+1})^k \frac{(q^{-j}; q)_k}{(aq^2; q)_k} p_{n-k}^* \left(x; aq^{k+1}, b|q \right); \tag{4.12}$$

$$d_{n-j}^n(x; a, b; q) = B_n (\sigma q^n)^j \frac{(q^{-n}/a; q)_j}{(bq; q)_j} \times \sum_{k=0}^{j} q^k \frac{(q^{-j}; q)_k}{(q^{-n}/a; q)_k} p_{n-k}^* \left(q^{k+1}x; a, bq^{k+1}|q \right), \tag{4.13}$$

where

$$A_n := \frac{(aq^2, \sigma q; q)_n}{C(aq)^n (q, bq; q)_n}, \qquad B_n := q^{-\binom{n+1}{2}} \frac{(\sigma q; q)_n}{C(-\sigma)^n (q; q)_n}. \tag{4.14}$$

Proof. Eqs. (4.12) and (4.13) follow by letting $\omega \uparrow 0$ in (3.21) and (3.22), respectively, and using (4.5) and (4.8). □

In particular, (4.12), (4.13) imply the formulas

$$d_0^n(x; a, b; q) = A_n p_n^*(x; aq, b|q); \qquad d_n^n(x; a, b; q) = B_n p_n^*(qx; a, bq|q),$$

where A_n and B_n are given in (4.14).

4.2.3. An identity

Corollary 4.5. *The following identity holds*:

$$\sum_{i=0}^{n} b_i^n(x; q)\, d_i^n(t; a, b; q) = L_n \frac{p_{n+1}^*(x) p_n^*(t) - p_{n+1}^*(t) p_n^*(x)}{x - t}, \tag{4.15}$$

where $p_k^*(x) = p_k^*(x; a, b|q)$, *and*

$$L_n := \frac{bq(aq; q)_{n+1}(\sigma q; q)_n}{Ca^n(\sigma q^{2n+1} - 1)(q, bq; q)_n}.$$

Proof. The result follows by letting $\omega \uparrow 0$ in (3.31), and using (3.3), (4.5) and (4.8). □

5. Dual Bernstein basis

5.1. Bernstein basis polynomials

Classical *Bernstein polynomial basis of degree n* ($n \in \mathbb{N}$) is given by (see, e.g., [3, p. 66])

$$B_i^n(x) := \binom{n}{i} x^i (1 - x)^{n-i} \quad (0 \leqslant i \leqslant n).$$

Recall that

$$R_k^{(\alpha,\beta)}(x) := \frac{(\alpha+1)_k}{k!}\,{}_2F_1\left(\left.\begin{matrix}-k,\,k+\alpha+\beta+1\\ \alpha+1\end{matrix}\,\right|\,1-x\right) \tag{5.1}$$

are the *shifted Jacobi polynomials* (see, e.g., [17, p. 280, Eq. (46)]). For $\alpha > -1$ and $\beta > -1$, polynomials (5.1) are orthogonal with respect to the scalar product [17, p. 273]

$$\langle f, g\rangle_J := \int_0^1 f(x)g(x)(1-x)^\alpha x^\beta\,\mathrm{d}x;$$

more specifically,

$$\left\langle R_k^{(\alpha,\beta)}, R_l^{(\alpha,\beta)}\right\rangle_J = K\frac{(\alpha+1)_k(\beta+1)_k}{k!(2k/\sigma+1)(\sigma)_k}\delta_{kl}, \tag{5.2}$$

where

$$K := \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\sigma+1)}, \quad \sigma := \alpha+\beta+1.$$

It can be shown that the orthogonality relation (4.2) reduces to (5.2) when $q \uparrow 1$, and we have [13, § 5.12]

$$\lim_{q\uparrow 1} p_k\left(x; q^\beta, q^\alpha\middle|q\right) = (-1)^k\frac{k!}{(\beta+1)_k}R_k^{(\alpha,\beta)}(x). \tag{5.3}$$

Recall that the *Hahn polynomials* are defined by [13, § 1.5]

$$Q_k(x;\alpha,\beta,N) := {}_3F_2\left(\left.\begin{matrix}-k,\,k+\alpha+\beta+1,\,-x\\ \alpha+1,\,-N\end{matrix}\,\right|\,1\right),$$

while the *dual Hahn polynomials* are given by [13, § 1.6]

$$R_k(\lambda(x);\gamma,\delta,N) := {}_3F_2\left(\left.\begin{matrix}-k,\,-x,\,x+\gamma+\delta+1\\ \gamma+1,\,-N\end{matrix}\,\right|\,1\right),$$

where $0 \leqslant k \leqslant N$, $N \in \mathbb{N}$, and $\lambda(x) := x(x+\gamma+\delta+1)$. Bernstein polynomials have the following representation in terms of shifted Jacobi polynomial basis:

$$B_i^n(x) = \binom{n}{i}(\alpha+1)_{n-i}(\beta+1)_i \times \sum_{j=0}^{n}\frac{(2j+\sigma)(-n)_j}{(\alpha+1)_j(j+\sigma)_{n+1}}Q_j(i;\beta,\alpha,n)\,R_j^{(\alpha,\beta)}(x), \tag{5.4}$$

where $0 \leqslant i \leqslant n$. This result follows from a formula given in [21], using [1, Corollary 3.3.5]. Conversely, shifted Jacobi polynomials have the following representation in terms of Bernstein basis [4]:

$$R_i^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_i}{i!}\sum_{j=0}^{n} R_{n-j}(\lambda(i);\alpha,\beta,n)\,B_j^n(x), \tag{5.5}$$

where $\lambda(i) := i(i+\sigma)$, $0 \leqslant i \leqslant n$.

5.2. Dual Bernstein basis polynomials

Dual Bernstein basis polynomials of degree n,

$$D_i^n(x;\alpha,\beta) \quad (i=0,1,\ldots,n),$$

are defined so that $\langle D_i^n, B_j^n\rangle_J = \delta_{ij}$ $(i,j=0,1,\ldots,n)$, where $D_i^n = D_i^n(\cdot;\alpha,\beta)$. In view of (3.1) and (3.3), and the observation preceding (5.3), we have

$$D_i^n(x;\alpha,\beta) = \lim_{q\uparrow 1} d_i^n(x;q^\beta,q^\alpha;q) \quad (0\leqslant i\leqslant n). \tag{5.6}$$

5.2.1. Recurrence relation

Limiting form of the recurrence relation (4.9) is given in the next theorem.

Theorem 5.1. *We have*

$$D_i^{n+1}(x;\alpha,\beta) = \left(1-\frac{i}{n+1}\right)D_i^n(x;\alpha,\beta) + \frac{i}{n+1}D_{i-1}^n(x;\alpha,\beta) + \vartheta_i^n R_{n+1}^{(\alpha,\beta)}(x), \tag{5.7}$$

where $0\leqslant i\leqslant n+1$, $D_{-1}^n(x;\alpha,\beta) = D_{n+1}^n(x;\alpha,\beta) = 0$, *and*

$$\vartheta_i^n := K^{-1}(-1)^{n+1-i}\frac{(2n+\sigma+2)(\sigma+1)_n}{(\beta+1)_i(\alpha+1)_{n+1-i}}.$$

Notice that for the special case $\alpha=\beta=0$ Eq. (5.7) reduces to a result obtained by Ciesielski [5].

5.2.2. Expansions

Theorem 5.2. *Shifted Jacobi polynomials have the following representation in terms of dual Bernstein basis*:

$$R_i^{(\alpha,\beta)}(x) = K\binom{n}{i}\frac{(\alpha+1)_i}{(\sigma+1)_{n+i}} \times\sum_{k=0}^{n}\binom{n}{k}(\alpha+1)_k(\beta+1)_{n-k}R_k(\lambda(i);\alpha,\beta,n)D_{n-k}^n(x;\alpha,\beta), \tag{5.8}$$

where $\lambda(i) := i(i+\sigma)$, $0\leqslant i\leqslant n$. *Dual Bernstein polynomials have the following representation in terms of the shifted Jacobi polynomial basis*:

$$D_i^n(x;\alpha,\beta) = K^{-1}\sum_{k=0}^{n}(-1)^k\frac{(2k/\sigma+1)(\sigma)_k}{(\alpha+1)_k}Q_k(i;\beta,\alpha,n)R_k^{(\alpha,\beta)}(x), \tag{5.9}$$

where $0\leqslant i\leqslant n$.

Proof. Set $a=q^\beta$, $b=q^\alpha$ in (4.10) and (4.11), respectively, let $q\uparrow 1$ and use (5.3), (3.1),

$$\lim_{q\uparrow 1} Q_m(q^{-x};q^A,q^B,N|q) = Q_m(x;A,B,N)$$

(see, e.g., [13, (5.6.1)]), and (5.6). □

Corollary 5.3. *We have*

$$D_i^n(x;\alpha,\beta) = D_{n-i}^n(1-x;\beta,\alpha) \quad (i=0,1,\ldots,n).$$

Proof. The result follows from (5.9) by using symmetry properties of shifted Jacobi and Hahn polynomials,

$$R_k^{(\alpha,\beta)}(x) = (-1)^k R_k^{(\beta,\alpha)}(1-x),$$
$$(\gamma+1)_k Q_k\,(x;\gamma,\delta,N) = (-1)^k(\delta+1)_k Q_k\,(N-x;\delta,\gamma,N)\,.$$

See, e.g., [1, p. 117], and [12]. □

Corollary 4.4 implies the following alternative formulas for the dual Bernstein polynomials.

Corollary 5.4. *The following formulas hold for* $i=0,1,\ldots,n$:

$$D_i^n(x;\alpha,\beta) = \frac{(-1)^{n-i}(\sigma+1)_n}{K(\alpha+1)_{n-i}(\beta+1)_i}\sum_{k=0}^{i}\frac{(-i)_k}{(-n)_k}R_{n-k}^{(\alpha,\beta+k+1)}(x), \tag{5.10}$$

$$D_{n-i}^n(x;\alpha,\beta) = \frac{(-1)^{i}(\sigma+1)_n}{K(\alpha+1)_{i}(\beta+1)_{n-i}}\sum_{k=0}^{i}(-1)^k\frac{(-i)_k}{(-n)_k}R_{n-k}^{(\alpha+k+1,\beta)}(x). \tag{5.11}$$

Notice that the coefficients of the linear combinations in (5.10) and (5.11) do not depend on α and β.

Proof. To obtain (5.10), set $a=q^\beta$, $b=q^\alpha$ in (4.12), let $q\uparrow 1$, then use (5.3), (3.1) and (5.6).
Formula (5.11) follows from (5.10) by using Corollary 5.3. □

In particular, we have

$$D_0^n(x;\alpha,\beta) = \frac{(-1)^n(\sigma+1)_n}{K(\alpha+1)_n}R_n^{(\alpha,\beta+1)}(x),$$
$$D_1^n(x;\alpha,\beta) = \frac{(-1)^{n-1}(\sigma+1)_n}{K(\alpha+1)_{n-1}(\beta+1)}\left[R_n^{(\alpha,\beta+1)}(x) + \frac{1}{n}R_{n-1}^{(\alpha,\beta+2)}(x)\right],$$
$$D_{n-1}^n(x;\alpha,\beta) = \frac{(\sigma+1)_n}{K(\alpha+1)(\beta+1)_{n-1}}\left[\frac{1}{n}R_{n-1}^{(\alpha+2,\beta)}(x) - R_n^{(\alpha+1,\beta)}(x)\right],$$
$$D_n^n(x;\alpha,\beta) = \frac{(\sigma+1)_n}{K(\beta+1)_n}R_n^{(\alpha+1,\beta)}(x).$$

5.2.3. An identity

As a limiting form of the identity (4.15) with $a=q^\beta$, $b=q^\alpha$ when $q\uparrow 1$, we obtain the following.

Corollary 5.5. *The following identity holds*:

$$\sum_{i=0}^{n} B_i^n(x)D_i^n(t;\alpha,\beta) = L_n\frac{R_{n+1}^{(\alpha,\beta)}(x)R_n^{(\alpha,\beta)}(t) - R_{n+1}^{(\alpha,\beta)}(t)R_n^{(\alpha,\beta)}(x)}{x-t},$$

where

$$L_n := \frac{(n+1)!(\sigma+1)_n}{K(2n+\sigma+1)(\alpha+1,\beta+1)_n}.$$

6. Applications

In computer-aided geometric design it is often necessary to find for a given function f the polynomial w_n of degree n that minimizes the error

$$d^2 := \langle f - w_n, f - w_n\rangle \tag{6.1}$$

(see, e.g., [7,8,15]). Obviously, the solution can be given in the form

$$w_n(x) = \sum_{k=0}^{n} a_k P_k(x), \tag{6.2}$$

where $a_k := \langle f, P_k\rangle/\langle P_k, P_k\rangle$ $(k = 0, 1, \ldots, n)$, and P_k's are orthogonal with respect to the given inner product. Note that the coefficients a_k do not depend on n. However, the preferred output representation should be compatible with CAGD conventions, e.g., the Bernstein–Bézier form. Furthermore, usually the degree n of the polynomial w_n is not known a priori. Instead, the degree is successively increased until the approximation error (6.1) diminishes below the prescribed value.

6.1. *Conversion of the orthogonal polynomial form to the Bézier form*

In this section, we discuss this topic in the most general context of Section 3. In the sequel, $P_k(x)$ denotes the big q-Jacobi polynomial $P_k(x; a, b, \omega/q; q)$ $(k\geqslant 0)$. Let w_n be a polynomial given by the formula (6.2), the coefficients a_k being known. According to (2.5), (3.11) and (3.19), the coefficients c_j^n in the generalized Bézier representation of w_n,

$$w_n(x) = \sum_{j=0}^{n} c_j^n B_j^n(x; \omega|q), \tag{6.3}$$

can be given by the formula:

$$c_j^n = T_n(q^{j-n}) \quad (j = 0, 1, \ldots, n),$$

where T_n is the following polynomial in q^{-x}:

$$T_n(q^{-x}) := \sum_{i=0}^{n} a_i Q_i(q^{-x}; a, b, n|q).$$

Notice that the values of $c_0^n, c_1^n, \ldots, c_n^n$ may be efficiently computed by using the Clenshaw's algorithm for evaluating a linear combination of orthogonal polynomials (see, e.g., [25, § 10.2.1]).

6.2. *Recursive construction of the Bézier form for the least-squares approximation polynomials*

Now, consider the problem of converting the polynomial

$$w_{n+1}(x) = \sum_{k=0}^{n+1} a_k P_k(x) = w_n(x) + a_{n+1} P_{n+1}(x)$$

to the generalized Bézier form

$$w_{n+1}(x) = \sum_{j=0}^{n+1} c_j^{n+1} B_j^{n+1}(x; \omega|q),$$

under the assumption that the Bézier coefficients c_i^n of the polynomial $w_n(x)$ are known (cf. (6.3)). Observing that

$$c_i^n = \langle f, D_i^n(\cdot; a, b, \omega|q)\rangle_{bqJ} \quad (i = 0, 1, \ldots, n),$$

and using recurrence relation (3.13), we obtain the formula

$$c_i^{n+1} = \gamma_i^n c_i^n + (1 - \gamma_{i-1}^n) c_{i-1}^n + \vartheta_i^n h_{n+1} a_{n+1} \quad (0 \leqslant i \leqslant n+1),$$

where $c_{-1}^n = c_{n+1}^n = 0$; the notation used is that of (3.6), (3.14), and (3.15).

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References

[1] G.E. Andrews, R. Askey, R. Roy, Special Functions, Cambridge University Press, Cambridge, 1999.

[2] I. Area, E. Godoy, P. Woźny, S. Lewanowicz, A. Ronveaux, Formulae relating q-Bernstein, little q-Jacobi and q-Hahn polynomials: application to the q-Bézier curve evaluation, Integral Transforms Spec. Funct. 15 (2004) 375–385.

[3] E.W. Cheney, Introduction to Approximation Theory, second ed., Chelsea Publishers, Providence, RI, 1982.

[4] Z. Ciesielski, Explicit formula relating the Jacobi, Hahn and Bernstein polynomials, SIAM J. Math. Anal. 18 (1987) 1573–1575.

[5] Z. Ciesielski, The basis of B-splines in the space of algebraic polynomials, Ukrainian Math. J. 38 (1987) 311–315.

[6] G.E. Farin, Curves and Surfaces for Computer-Aided Geometric Design: A Practical Guide, third ed., Academic Press, Boston, 1996.

[7] R.T. Farouki, Legendre–Bernstein basis transformations, J. Comput. Appl. Math. 119 (2000) 145–160.

[8] R.T. Farouki, Convergent inversion approximations for polynomials in Bernstein form, Comput. Aided Geom. Design 17 (2000) 179–196.

[9] G. Gasper, M. Rahman, Basic Hypergeometric Series, second ed., Cambridge University Press, Cambridge, 2004.

[10] R.N. Goldman, Dual polynomial bases, J. Approx. Theory 79 (1994) 311–346.

[11] B. Jüttler, The dual basis functions of the Bernstein polynomials, Adv. Comput. Math. 8 (1998) 345–352.

[12] S. Karlin, J. McGregor, The Hahn polynomials, formulas, and an application, Scripta Math. 26 (1961) 33–46.

[13] R. Koekoek, R.F. Swarttouw, The Askey scheme of hypergeometric orthogonal polynomials and its q-analogue, Report No. 98–17, Fac. Techn. Math. Informatics, Delft University of Technology, Delft, 1998.

[14] T.H. Koornwinder, Compact quantum groups and q-special functions, in: V. Baldoni, M.A. Picardello (Eds.), Representations of Lie Groups and Quantum Groups, Pitman Research Notes in Mathematics Series, vol. 311, Longman Scientific & Technical, 1994, pp. 46–128.

[15] B.-G. Lee, Y. Park, J. Yoo, Application of Legendre–Bernstein basis transformations to degree elevation and degree reduction, Comput. Aided Geom. Design 19 (2002) 709–718.

[16] S. Lewanowicz, P. Woźny, Generalized Bernstein polynomials, BIT 44 (2004) 63–78.
[17] Y.L. Luke, The Special Functions and Their Approximations, vol. 1, Academic Press, New York, 1969.
[18] H. Oruç, G.M. Phillips, q-Bernstein polynomials and Bézier curves, J. Comput. Appl. Math. 151 (2003) 1–12.
[19] G.M. Phillips, A de Casteljau algorithm for generalized Bernstein polynomials, BIT 36 (1996) 232–236.
[20] G.M. Phillips, Bernstein polynomials based on the q-integers, Ann. Numer. Math. 4 (1997) 511–518.
[21] A. Ronveaux, A. Zarzo, I. Area, E. Godoy, Bernstein bases and Hahn–Eberlein orthogonal polynomials, Integral Transforms Spec. Funct. 7 (1998) 87–96.
[22] P. Sablonnière, Discrete Bézier curves and surfaces, in: T. Lyche, L.L. Schumaker (Eds.), Mathematical Methods in Computer Aided Geometric Design II, Academic Press, New York, 1992, pp. 497–515.
[23] P. Sablonnière, Discrete Bernstein bases and Hahn polynomials, J. Comput. Appl. Math. 49 (1993) 233–241.
[24] D. Stanton, Product formulas for q-Hahn polynomials, SIAM J. Math. Anal. 11 (1980) 100–107.
[25] J. Wimp, Computation with Recurrence Relations, Pitman, Boston, 1984.